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A section theorem with applications to coincidence theorems and minimax inequalities in *FWC*-spaces

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ABSTRACT

In this paper, an improved version of section theorem is proved in *FWC*-spaces without any linear and convex structure under much weaker assumptions, and next as its applications, some new coincidence theorems and minimax inequalities are established in *FWC*-spaces. These results generalize many known theorems in the literature.

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1. Introduction

In 1961, Fan [1] generalized the classical *KKM* theorem from finite dimensional spaces to infinite dimensional Hausdorff topological vector spaces and established the following section theorem.

Theorem A. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space E and let $A \subset X \times X$ be a subset such that*

(a) *for each $y \in X$, the set $\{x \in X : (x, y) \in A\}$ is closed in X;*
(b) *for each $x \in X$, the set $\{y \in X : (x, y) \notin A\}$ is convex or empty;*
(c) *for each $x \in X$, $(x, x) \in A$.*

Then there exists a point $x_0 \in X$ such that $\{x_0\} \times X \subset A$.

Since then, a number of generalizations with their applications have been studied by many authors (see, for example, [2–6] and references therein). Lin et al. [7] proved some section theorems in the setting of convex spaces. By using these results, they derived some coincidence theorems, fixed point theorems and some existence results for a solution to the generalized vector equilibrium problems under suitable assumptions. Lan [8] proved an intersection theorem and obtained some applications to the existence of maximal and greatest elements for strict and weak relations, a section theorem and some minimax inequalities in topological vector spaces. Recently, Balaj and Lin [9] have established some existence theorems of solutions for variational relation problems in convex spaces with applications to fixed point theorems, generalized maximal element theorems, a generalized coincidence theorem and a section theorem.

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It is well known that the linear and convex assumptions play a crucial role in most of known section theorems, which strictly restricts the applicable area of these section theorems. Hence, Wu and Li [10] proved a very general section theorem and applied it to minimax inequalities and coincidence theorems in H-spaces. Balaj [11] obtained some section theorems and gave their applications to minimax inequalities in G-convex spaces. Recently, Ding [12] has studied *KKM* theory for the set-valued mapping and obtained some section theorems in *FC*-spaces.

Motivated and inspired by the recent work on section theorems, in this paper, we first introduce the concept of an *FWC*-space without any linear and convex structure, and next, we give an improved version of section theorem in *FWC*-spaces. As applications of this section theorem, some new coincidence theorems and minimax inequalities are obtained in *FWC*-spaces under much weaker assumptions. These results generalize and unify many known theorems in the literature.

2. Preliminaries

The set of all real numbers is denoted by $\mathbb{R}$ and the set of natural numbers is denoted by $\mathbb{N}$. Let X be a set. We shall denote by 2^X the family of all subsets of X, by $\langle X\rangle$ the family of nonempty finite subsets of X. For any $A \in \langle X\rangle$, we shall denote by $|A|$ the cardinality of A. Let A be a subset of a topological space X. int A and $\overline{A}$ stand for the interior and the closure of A, respectively. If A is a subset of a vector space, we shall denote by co A the convex hull of A. Let Δ_n denote the standard n-dimensional simplex with vertices $\{e_0, e_1, \dots, e_n\}$. For a nonempty subset $J \subset \{0, 1, \dots, n\}$, let $\triangle_{|J|-1}$ denote the convex hull of the vertices $\{e_j : j \in J\}$.

Let X and Y be two nonempty sets and let $T : X \to 2^Y$ be a set-valued mapping. Then the set-valued mappings $T^{-1} : Y \to 2^X$ and $T^* : Y \to 2^X$ are defined by $T^{-1}(y) = \{x \in X : y \in T(x)\}$ and $T^*(y) = X \setminus T^{-1}(y)$ for each $y \in Y$, respectively. $T^c : X \to 2^Y$ is defined by $T^c(x) = Y \setminus T(x)$ for each $x \in X$. Let $T(X_0) = \cup_{x\in X_0} T(x)$ for any $X_0 \subset X$. Note that the graph of T, denoted by Graph (T), is the set $\{(x, y) \in X \times Y : y \in T(x)\}$. Given two set-valued mappings $T : X \to 2^Y$ and $S : Y \to 2^Z$, the composition $S \circ T : X \to 2^Z$ is defined by $(S \circ T)(x) = S(T(x)) = \cup\{S(y) : y \in T(x)\}$ for each $x \in X$. Let X be a topological space and Y be a nonempty set. A set-valued mapping $T : Y \to 2^X$ is said to be compact if $T(Y) \subset K$ for some compact subset K of X.

A subset A of a topological space X is said to be compactly closed (resp., compactly open) in X if for each nonempty compact subset $C \subset X$, $A \cap C$ is closed (resp., open) in C. The compact closure and the compact interior of A (see [13]) are defined by

$$\operatorname{ccl} A = \bigcap\{B : A \subset B \text{ and } B \text{ is compactly closed in } X\}, \text{ and}$$
$$\operatorname{cint} A = \bigcup\{B : B \subset A \text{ and } B \text{ is compactly open in } X\},$$

respectively. It is easy to see that ccl A (resp., cint A) is compactly closed (resp., compactly open) in X and A is compactly closed (resp., compactly open) if and only if $A = \operatorname{ccl} A$ (resp., $A = \operatorname{cint} A$). For each nonempty compact subset C of X, we have $(\operatorname{ccl} A) \cap C = \operatorname{cl}_C (A \cap C)$ and $(\operatorname{cint} A) \cap C = \operatorname{int}_C (A \cap C)$, where $\operatorname{cl}_C (A \cap C)$ and $\operatorname{int}_C (A \cap C)$ denote the closure and the interior of $A \cap C$ in C, respectively.

Definition 2.1 (*[13]*)**.** Let X be a topological space, Y be a nonempty set, and let $F : Y \to 2^X$ be a set-valued mapping. F is said to be transfer compactly closed-valued (resp., transfer compactly open-valued) on Y if for each $y \in Y$ and for each nonempty compact subset C of X, $x \notin F(y) \cap C$ (resp., $x \in F(y) \cap C$) implies that there exists $y' \in Y$ such that $x \notin \operatorname{cl}_C(F(y') \cap C)$ (resp., $x \in \operatorname{int}_C(F(y') \cap C)$).

By the above definition, we can easily verify that F is transfer compactly closed-valued if and only if F^c is transfer compactly open-valued.

Definition 2.2 (*[13]*)**.** Let X be a topological space, Y be a nonempty set, and $f : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be a function. For some $\lambda \in \mathbb{R}$, $f(x, y)$ is said to be λ-transfer compactly lower (resp., upper) semicontinuous on X if for each compact subset C of X and for each $x \in X, y \in Y$ with $f(x, y) > \lambda$ (resp., $f(x, y) < \lambda$), there exist a relatively open neighborhood $\mathcal{O}(x)$ of x in C and a point $y' \in Y$ such that $f(z, y') > \lambda$ (resp., $f(z, y') < \lambda$) for all $z \in \mathcal{O}(x)$. If f is λ-transfer compactly lower (resp., upper) semicontinuous on X for each $\lambda \in \mathbb{R}$, we say that f is transfer compactly lower (resp., upper) semicontinuous on X.

Lemma 2.1 (*[13]*)**.** *Let X, Y, and f be the same as in Definition 2.2. For some $\lambda \in \mathbb{R}$, f is λ-transfer compactly lower (resp., upper) semicontinuous on X if and only if the set-valued mapping $F : Y \to 2^X$ defined by $F(y) = \{x \in X : f(x, y) \le \lambda\}$ (resp., $F(y) = \{x \in X : f(x, y) \ge \lambda\}$) for each $y \in Y$, is transfer compactly closed-valued.*

Definition 2.3. A triple $(Y, D; \varphi_N)$ is said to be a finite weakly convex space (shortly, an *FWC*-space) if Y, D are two nonempty sets and for each $N = \{u_0, \dots, u_n\} \in \langle D\rangle$ where some elements in N may be same, there exists a set-valued mapping $\varphi_N : \Delta_n \to 2^Y$ with nonempty values. When $D \subset Y$, the space is denoted by $(Y \supset D; \varphi_N)$. In case $Y = D$, let $(Y; \varphi_N) := (Y, Y; \varphi_N)$.

Example 2.1. For any $r \in \mathbb{R}$, let $[r]$ denote the integer part of r. Let $D = (0, 1)$ and $Y = \mathbb{N} \cup (-\mathbb{N})$. For each $N = \{u_0, \ldots, u_n\} \in \langle D \rangle$, define a set-valued mapping $\varphi_N : \Delta_n \to 2^Y$ by

$$\varphi_N(z) = \left\{ \pm \max \left\{ \left[\frac{1}{z_j u_j} \right] : j \in J(z) \right\}, \pm \min \left\{ \left[\frac{1}{z_j u_j} \right] : j \in J(z) \right\} \right\} \quad \text{for all } z \in \Delta_n,$$

where $z = \sum_{i=0}^{n} z_i e_i \in \Delta_n$ and $J(z) := \{j \in \{0, 1, \ldots, n\} : z_j \neq 0\}$. It is easy to see that $\varphi_N(z) \neq \emptyset$ for each $z \in \Delta_n$. Then $(Y, D; \varphi_N)$ forms an *FWC*-space.

Let $A \subset D$ and $B \subset Y$. B is said to be an *FWC*-subspace of $(Y, D; \varphi_N)$ relative to A if for each $N = \{u_0, \ldots, u_n\} \in \langle D \rangle$ and for each $\{u_{i_0}, \ldots, u_{i_k}\} \subset A \cap \{u_0, \ldots, u_n\}$, we have $\varphi_N(\Delta_k) \subset B$, where $\Delta_k = \mathrm{co}(\{e_{i_0}, \ldots, e_{i_k}\})$. We note that if A is nonempty and B is an *FWC*-subspace of $(Y, D; \varphi_N)$ relative to A, then B is automatically nonempty. When $A = B$, B is said to be an *FWC*-subspace of $(Y \supset D; \varphi_N)$. We set a rule that $\emptyset$ is an *FWC*-subspace of $(Y, D; \varphi_N)$ relative to $\emptyset$.

It is worthwhile noticing that Y and D in Definition 2.3 do not possess any linear, convex and topological structure and so the set-valued mapping φ_N has no continuity requirement. Even Y is a topological space, it is easy to see that convex subsets of topological vector spaces, Lassonde's convex spaces in [14], H-spaces introduced by Horvath [15], G-convex spaces introduced by Park and Kim [16], L-convex spaces introduced by Ben-El-Mechaiekh et al. [17], G-H-spaces introduced by Verma [18,19], pseudo H-spaces introduced by Lai et al. [20], *GFC*-spaces due to Khanh et al. [21], *FC*-spaces due to Ding [22], and many other topological spaces with abstract convex structure (see, for example, [23–25] and references therein) are all particular forms of *FWC*-spaces. Hence, it is quite reasonable and valuable to study various nonlinear problems in *FWC*-spaces.

Definition 2.4. Let X be a topological space and $(Y, D; \varphi_N)$ be an *FWC*-space. The class $\widetilde{\mathcal{B}}(Y, D, X)$ of better admissible mappings is defined as follows: a set-valued mapping $T : Y \to 2^X$ belongs to $\widetilde{\mathcal{B}}(Y, D, X)$ if and only if for any $N = \{u_0, \ldots, u_n\} \in \langle D \rangle$ and for any continuous mapping $\psi : T(\varphi_N(\Delta_n)) \to \Delta_n$, the composition $\psi \circ T|_{\varphi_N(\Delta_n)} \circ \varphi_N : \Delta_n \to 2^{\Delta_n}$ has a fixed point. When $Y = D$, we shall write $\widetilde{\mathcal{B}}(Y, X)$ instead of $\widetilde{\mathcal{B}}(Y, D, X)$.

Since Y and D in Definition 2.4 are two nonempty sets which do not possess any linear, convex and topological structure, the class $\widetilde{\mathcal{B}}(Y, D, X)$ unifies and extends many important classes of mappings, for example, the class $\mathcal{U}_C^K(X, Y)$ in [16], the class $A(Y, X)$ in [17] and the class $\mathcal{B}(Y, X)$ in [22].

From now on, all topological spaces are assumed to be Hausdorff unless otherwise specified.

3. A section theorem

We begin with a new section theorem in *FWC*-spaces, which is needed in this paper.

Theorem 3.1. *Let X be a topological space, $(Y, D; \varphi_N)$ be an FWC-space, and let $T \in \widetilde{\mathcal{B}}(Y, D, X)$ be a compact set-valued mapping. Let $P \subset L \subset X \times D$ and $M \subset X \times Y$ such that*

(i) *the set-valued mapping $A : D \to 2^X$ is transfer compactly open-valued, where A is defined by $A(u) = \{x \in X : (x, u) \notin L\}$ for each $u \in D$;*
(ii) *for each $x \in \overline{T(Y)}$, the set $\{y \in Y : (x, y) \notin M\}$ is an FWC-subspace of $(Y, D; \varphi_N)$ relative to the set $\{u \in D : (x, u) \notin P\}$;*
(iii) $\mathrm{Graph}(T^{-1}) \subset M$.

Then there exists a point $\hat{x} \in \overline{T(Y)}$ such that $\{\hat{x}\} \times D \subset L$.

Proof. Suppose that the conclusion of Theorem 3.1 is false. Then for each $x \in \overline{T(Y)}$, there exists $u \in D$ such that $(x, u) \notin L$, i.e., $x \in A(u) \cap \overline{T(Y)}$. By (i), there exists $u' \in D$ such that $x \in \mathrm{int}_{\overline{T(Y)}}(A(u') \cap \overline{T(Y)}) = (\mathrm{cint}\, A(u')) \cap \overline{T(Y)}$. Hence, we have

$$\overline{T(Y)} \subset \bigcup \{(\mathrm{cint}\, A(u)) \cap \overline{T(Y)} : u \in D\}.$$

Since each $\mathrm{cint}\, A(u)$ is compactly open and the set $\overline{T(Y)}$ is compact, there exists $N = \{u_0, \ldots, u_n\} \in \langle D \rangle$ such that

$$\overline{T(Y)} \subset \bigcup \{(\mathrm{cint}\, A(u)) \cap \overline{T(Y)} : u \in N\}.$$

Let $\{\alpha_0, \ldots, \alpha_n\}$ be a partition of unity on $\overline{T(Y)}$ subordinated to the open covering $\{(\mathrm{cint}\, A(u)) \cap \overline{T(Y)} : u \in N\}$, which means that

$$\begin{cases} \alpha_i : \overline{T(Y)} \to [0, 1] \quad \text{is continuous for each } i \in \{0, 1, \ldots, n\}; \\ \alpha_i(x) > 0 \Rightarrow x \in (\mathrm{cint}\, A(u_i)) \cap \overline{T(Y)}; \\ \displaystyle\sum_{i=0}^{n} \alpha_i(x) = 1 \quad \text{for each } x \in \overline{T(Y)}. \end{cases}$$

Define a function $\psi : \overline{T(Y)} \to \Delta_n$ by $\psi(x) = \sum_{i=0}^{n} \alpha_i(x)e_i$ for each $x \in \overline{T(Y)}$. Clearly, ψ is continuous and for each $x \in \overline{T(Y)}$, we have $\psi(x) = \sum_{i\in J(x)} \alpha_i(x)e_i \in \Delta_{|J(x)|-1}$, where $J(x) = \{i \in \{0, 1, \ldots, n\} : \alpha_i(x) > 0\}$. Since $P \subset L$, we have $x \in (\mathrm{cint}\, A(u_i)) \cap \overline{T(Y)} \subset \{x \in X : (x, u_i) \notin P\}$ for each $i \in J(x)$; hence, $\{u_i : i \in J(x)\} \subset N \cap \{u \in D : (x, u) \notin P\}$. It follows from the definition of an *FWC*-space that there exists a set-valued mapping $\varphi_N : \Delta_n \to 2^Y$ with nonempty values. Now, we define a set-valued mapping $f : \overline{T(Y)} \to 2^Y$ as follows:

$$f(x) = \varphi_N(\psi(x)) \quad \text{for each } x \in \overline{T(Y)}. \tag{3.1}$$

By the definition of ψ and (ii), we have

$$f(x) = \varphi_N(\psi(x)) \subset \varphi_N(\Delta_{|J(x)|-1}) \subset \{y \in Y : (x, y) \notin M\} \quad \text{for all } x \in \overline{T(Y)}.$$

This shows that

$$(x, y) \notin M \quad \text{for all } x \in \overline{T(Y)} \text{ and all } y \in f(x). \tag{3.2}$$

On the other hand, since $T \in \widetilde{\mathfrak{B}}(Y, D, X)$, the composition $\psi \circ T|_{\varphi_N(\Delta_n)} \circ \varphi_N : \Delta_n \to 2^{\Delta_n}$ has a fixed point $\bar{z} \in \Delta_n$, i.e., $\bar{z} \in \psi \circ T|_{\varphi_N(\Delta_n)} \circ \varphi_N(\bar{z})$. Hence, there exists $\bar{x} \in T|_{\varphi_N(\Delta_n)} \circ \varphi_N(\bar{z})$ such that $\bar{z} = \psi(\bar{x})$. Choose $\bar{y} \in \varphi_N(\bar{z})$ such that $\bar{x} \in T|_{\varphi_N(\Delta_n)}(\bar{y})$. By (3.1) and (iii), we have

$$\bar{y} \in \varphi_N(\bar{z}) = \varphi_N(\psi(\bar{x})) = f(\bar{x}) \quad \text{and} \quad (\bar{x}, \bar{y}) \in \mathrm{Graph}(T^{-1}) \subset M,$$

which contradicts (3.2). Thus, there must exist a point $\hat{x} \in \overline{T(Y)}$ such that $\{\hat{x}\} \times D \subset L$. This completes the proof. □

Remark 3.1. Theorem 3.1 generalizes Theorem 2 of [6] and Theorem 4.4 of [9] from topological vector spaces to *FWC*-spaces without any linear and convex structure under much weaker assumptions. Theorem 3.1 also generalizes Theorem 1 of [10] from *H*-spaces to *FWC*-spaces.

We give the following example, which shows that all the conditions of Theorem 3.1 hold under the setting of *FWC*-spaces.

Example 3.1. Let $X = (-1, +\infty)$ be endowed with Euclidean topology. Let $D = [-4, 0]$ and $(Y, \tau) = ([0, 4], \{[0, 4], \emptyset, [0, 1], [1, 2], [2, 3], [3, 4]\})$, where τ is a family of subsets of Y. It is easy to check that (Y, τ) is not a topological space. For simplicity, we shall write Y instead of (Y, τ). Let $M \subset X \times Y$ and $P = L \subset X \times D$ be given by

$$\begin{aligned} M \;=\; & \{(x, y) \in E_1 : y \le x + 2\} \\ & \bigcup \{(x, y) \in E_2 : -x + 2 \le y \le x + 2\} \\ & \bigcup \{(x, y) \in E_3 : y \ge -x + 2\}, \end{aligned}$$

$$P = L = \{(x, u) \in E_4 : u \le x - 2\} \bigcap \{(x, u) \in E_4 : u \ge -x - 2\},$$

where $E_1 = \{(x, y) \in X \times Y : 0 \le x \le 3, 3 \le y \le 4\}$, $E_2 = \{(x, y) \in X \times Y : 0 \le x \le 1, 1 \le y \le 3\}$, $E_3 = \{(x, y) \in X \times Y : 0 \le x \le 3, 0 \le y \le 1\}$ and $E_4 = \{(x, u) \in X \times D : 0 \le x \le 3, -4 \le u \le 0\}$. For any $u \in D$, we have

$$\begin{aligned} A(u) &= \{x \in X : (x, u) \notin L\} \\ &= \begin{cases} (-1, -u - 2) \bigcup (3, +\infty), & \text{if } -4 \le u \le -2, \\ (-1, u + 2) \bigcup (3, +\infty), & \text{if } -2 < u \le 0, \end{cases} \end{aligned}$$

which is open in X. Therefore, A is transfer compactly open-valued, and hence, Condition (1) of Theorem 3.1 is satisfied. Now, let $T : Y \to 2^X$ be a set-valued mapping defined by

$$T(y) = \begin{cases} [2, 3], & \text{if } 0 \le y < 1, \\ [1, 3], & \text{if } y = 1, \\ \{1\}, & \text{if } 1 < y < 3, \\ [1, 3], & \text{if } y = 3, \\ [2, 3], & \text{if } 3 < y \le 4. \end{cases}$$

Obviously, $\overline{T(Y)} = [1, 3]$, which is compact subset of X. In order to check Condition (ii) of Theorem 3.1, for any finite set $N = \{u_0, \ldots, u_n\} \in \langle D\rangle$, we define a set-valued mapping $\varphi_N : \Delta_n \to 2^Y$ by $\varphi_N(z) = \{0, 4\}$ for all $z \in \Delta_n$. Then $(Y, D; \varphi_N)$ is an *FWC*-space, which is not a *GFC*-space since φ_N is a set-valued mapping and Y is not a topological space. For each $x \in [1, 3]$, we have

$$\{y \in Y : (x, y) \notin M\} = \begin{cases} [0, 1) \bigcup (3, 4], & \text{if } x = 1, \\ [0, -x + 2) \bigcup (x + 2, 4] \bigcup (1, 3), & \text{if } 1 < x < 2, \\ (1, 3), & \text{if } 2 \le x \le 3. \end{cases}$$

For each $x \in [1, 2)$, the set $\{u \in D : (x, u) \notin P\} = (x-2, 0]\bigcup[-4, -x-2) \neq \emptyset$. Therefore, for each $x \in [1, 2)$, each $N = \{u_0, \ldots, u_n\} \in \langle D\rangle$ and each $\{u_{i_0}, \ldots, u_{i_k}\} \subset N \cap \{u \in D : (x, u) \notin P\}$, we have $\varphi_N(\Delta_k) = \{0, 4\} \subset \{y \in Y : (x, y) \notin M\}$. Since the set $\{u \in D : (x, u) \notin P\} = \emptyset$ for each $x \in [2, 3]$, it follows that the set $\{y \in Y : (x, y) \notin M\}$ is an *FWC*-subspace of $(Y, D; \varphi_N)$ relative to the set $\{u \in D : (x, u) \notin P\}$ automatically. Thus, Condition (ii) of Theorem 3.1 is fulfilled. Finally, we check that $\mathrm{Graph}(T^{-1}) \subset M$ and $T \in \widetilde{\mathfrak{B}}(Y, D, X)$. By the definition of T, we can obtain $T^{-1} : X \to 2^Y$, which is given as follows, for each $x \in X$,

$$T^{-1}(x) = \{0 \le y \le 4 : x \in T(y)\}$$
$$= \begin{cases} \emptyset, & \text{if } -1 < x < 1, \\ [1, 3], & \text{if } x = 1, \\ \{1, 3\}, & \text{if } 1 < x < 2, \\ [3, 4]\bigcup[0, 1], & \text{if } 2 \le x \le 3, \\ \emptyset, & \text{if } 3 < x < +\infty. \end{cases}$$

Thus, we can easily see that $\mathrm{Graph}(T^{-1}) \subset M$. For each $N = \{u_0, \ldots, u_n\} \in \langle D\rangle$ and each $z \in \Delta_n$, we know that $T(\varphi_N(z)) = T(0) \cup T(4) = [2, 3]$, which is nonempty compact convex and so contractible subset of X. Then it follows that the composition $T|_{\varphi_N(\Delta_n)} \circ \varphi_N : \Delta_n \to 2^{T(\varphi_N(\Delta_n))}$ is an upper semicontinuous set-valued mapping with nonempty compact contractible values. By Lemma 1 in [6], for any single-valued continuous mapping $\psi : T(\varphi_N(\Delta_n)) = [2, 3] \to \Delta_n$, the composition $\psi \circ T|_{\varphi_N(\Delta_n)} \circ \varphi_N : \Delta_n \to 2^{\Delta_n}$ has a fixed point. Hence, $T \in \widetilde{\mathfrak{B}}(Y, D, X)$. Therefore, all the hypotheses of Theorem 3.1 are satisfied. By direct checking, we see that there exists a point $\widehat{x} = \frac{5}{2} \in \overline{T(Y)}$ such that $\{\frac{5}{2}\} \times D \subset L$.

Note that in Theorem 3.1, we can see that the assumption that for each $x \in \overline{T(Y)}$, the set $\{y \in Y : (x, y) \notin M\}$ is an *FWC*-subspace of $(Y, D; \varphi_N)$ relative to the set $\{u \in D : (x, u) \notin P\}$ is essential from the following example:

Example 3.2. Let X, Y, D, M, and T be the same as in Example 3.1. Let $P = L \subset X \times D$ be defined by $P = L = \{(x, u) \in E_5 : u \le x - 2\} \bigcap \{(x, u) \in E_5 : u \ge -x - 2\}$, where $E_5 = \{(x, u) \in X \times D : 0 \le x \le 3, -4 \le u \le -1\}$. For any $u \in D$, we have

$$A(u) = \{x \in X : (x, u) \notin L\}$$
$$= \begin{cases} (-1, -u-2)\bigcup(3, +\infty), & \text{if } -4 \le u \le -2, \\ (-1, u+2)\bigcup(3, +\infty), & \text{if } -2 < u \le -1, \\ X, & \text{if } -1 < u \le 0, \end{cases}$$

which is open in X. Therefore, A is transfer compactly open-valued. In Example 3.1, we have checked that T is compact and $\mathrm{Graph}(T^{-1}) \subset M$. Now, suppose that there exists an *FWC*-space $(Y, D; \varphi_N)$ such that $T \in \widetilde{\mathfrak{B}}(Y, D, X)$. In fact, the *FWC*-space defined in Example 3.1 satisfies this requirement. For each $x \in \overline{T(Y)} = [1, 3]$, we have

$$\{u \in D : (x, u) \notin L\} = \begin{cases} (-1, 0]\bigcup[-4, -x-2), & \text{if } 1 \le x < 2, \\ (-1, 0], & \text{if } 2 \le x \le 3. \end{cases}$$

Then it follows that for each $x \in \overline{T(Y)}$, there exists $u_0 \in D$ such that $(x, u_0) \notin L$, which implies that the conclusion of section theorem cannot be true. Thus, the assumption that for each $x \in \overline{T(Y)}$, the set $\{y \in Y : (x, y) \notin M\}$ is an *FWC*-subspace of $(Y, D; \varphi_N)$ relative to the set $\{u \in D : (x, u) \notin P\}$ does not hold. Suppose the contrary. Then we may take $x_1 = 1 \in \overline{T(Y)}$ and $x_2 = 2 \in \overline{T(Y)}$. For the case $n = 5$, we let $N = \{-\frac{1}{5}, -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}, -1, -2\} \in \langle D\rangle$. Since $\{-\frac{1}{5}, -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}\} \subset \{u \in D : (1, u) \notin P\} \cap \{u \in D : (2, u) \notin P\}$, we have $\varphi_N(\Delta_3) \subset [0, 1)\bigcup(3, 4]$ and $\varphi_N(\Delta_3) \subset (1, 3)$, which is a contradiction. Hence, the assumption that for each $x \in \overline{T(Y)}$, the set $\{y \in Y : (x, y) \notin M\}$ is an *FWC*-subspace of $(Y, D; \varphi_N)$ relative to the set $\{u \in D : (x, u) \notin P\}$ does not hold.

4. Coincidence theorems

From Theorem 3.1, we obtain the following generalized coincidence theorem.

Theorem 4.1. *Let X be a topological space, $(Y, D; \varphi_N)$ be an FWC-space, Z be a nonempty set, and let $T \in \widetilde{\mathfrak{B}}(Y, D, X)$ be a compact set-valued mapping. Suppose $F : D \to 2^X$, $R : X \to 2^D$, $\Omega : X \to 2^Z$, and $Q : Y \to 2^Z$ are set-valued mappings such that*

(i) *$F^c : D \to 2^X$ is transfer compactly open-valued;*
(ii) *for each $u \in D$, $\{x \in X : u \notin R(x)\} \subset F(u)$;*
(iii) *for each $x \in \overline{T(Y)}$, the set $\{y \in Y : \Omega(x) \cap Q(y) \neq \emptyset\}$ is an FWC-subspace of $(Y, D; \varphi_N)$ relative to $R(x)$;*
(iv) *for each $x \in \overline{T(Y)}$, $F^*(x) \neq \emptyset$.*

Then there exists $(\widehat{x}, \widehat{y}) \in X \times Y$ such that $\widehat{x} \in T(\widehat{y})$ and $\Omega(\widehat{x}) \cap Q(\widehat{y}) \neq \emptyset$.

Proof. Let $L = \{(x, u) \in X \times D : x \in F(u)\}$, $M = \{(x, y) \in X \times Y : \Omega(x) \cap Q(y) = \emptyset\}$ and $P = \{(x, u) \in X \times D : u \notin R(x)\}$. Define a set-valued mapping $A : D \to 2^X$ by

$$A(u) = \{x \in X : (x, u) \notin L\} \quad \text{for each } u \in D.$$

It is easy to see that $A(u) = F^c(u)$ for each $u \in D$. Then by (i), A is transfer compactly open-valued. By (ii), we have $P \subset L$. By (iii), for each $x \in \overline{T(Y)}$, the set $\{y \in Y : (x, y) \notin M\}$ is an *FWC*-subspace of $(Y, D; \varphi_N)$ relative to the set $\{u \in D : (x, u) \notin P\}$. Suppose the conclusion of Theorem 4.1 is false. Then for each $y \in Y$ and each $x \in T(y)$, we have $\Omega(x) \cap Q(y) = \emptyset$, which implies that $\{(x, y) \in X \times Y : x \in T(y)\} \subset \{(x, y) \in X \times Y : \Omega(x) \cap Q(y) = \emptyset\}$, i.e., $\mathrm{Graph}(T^{-1}) \subset M$. So, it follows from Theorem 3.1 that there exists a point $\widehat{x} \in \overline{T(Y)}$ such that $\{\widehat{x}\} \times D \subset L$, i.e., $\widehat{x} \in F(u)$ for all $u \in D$, which implies that $F^*(\widehat{x}) = \emptyset$. This contradicts (iv). Therefore, there exists $(\widehat{x}, \widehat{y}) \in X \times Y$ such that $\widehat{x} \in T(\widehat{y})$ and $\Omega(\widehat{x}) \cap Q(\widehat{y}) \neq \emptyset$. The proof is completed. $\square$

Remark 4.1. Theorem 4.1 generalizes Corollary 6 of [26] in the following aspects: (a) from *G*-convex spaces to *FWC*-spaces without any linear and convex structure; (b) from $\mathcal{B}(X, Y)$ to $\widetilde{\mathcal{B}}(Y, D, X)$; (c) Conditions (ii) and (iii) of Corollary 6 in [26] are dropped; (d) Condition (iii) of Theorem 4.1 is weaker than Condition (iv) of Corollary 6 in [26].

When $Y = Z$ and $Q(y) = \{y\}$ for each $y \in Y$, Theorem 4.1 reduces to the following coincidence theorem.

Theorem 4.2. *Let X be a topological space, $(Y, D; \varphi_N)$ be an FWC-space, and let $T \in \widetilde{\mathcal{B}}(Y, D, X)$ be a compact set-valued mapping. Suppose $F : D \to 2^X$, $R : X \to 2^D$, and $\Omega : X \to 2^Y$ are set-valued mappings such that*

(i) *$F^c : D \to 2^X$ is transfer compactly open-valued;*
(ii) *for each $u \in D$, $\{x \in X : u \notin R(x)\} \subset F(u)$;*
(iii) *for each $x \in \overline{T(Y)}$, $\Omega(x)$ is an FWC-subspace of $(Y, D; \varphi_N)$ relative to $R(x)$;*
(iv) *for each $x \in \overline{T(Y)}$, $F^*(x) \neq \emptyset$.*

Then there exists $(\widehat{x}, \widehat{y}) \in X \times Y$ such that $\widehat{x} \in T(\widehat{y})$ and $\widehat{y} \in \Omega(\widehat{x})$.

Remark 4.2. Theorem 4.2 generalizes Theorem 7 of [10] in the following aspects: (a) from *H*-spaces to *FWC*-spaces; (b) Conditions (i) and (iv) of Theorem 7 in [10] are dropped; (c) Conditions (i) and (iv) of Theorem 4.2 are weaker than Condition (ii) of Theorem 7 in [10]; (d) Condition (iii) of Theorem 4.2 is weaker than Condition (iii) of Theorem 7 in [10]; (e) from two set-valued mappings to three set-valued mappings; in turn, it improves and generalizes Theorem 1 of [27], Theorem 1 of [28], and Theorem 2.3 of [29] from topological vector spaces to *FWC*-spaces without any linear and convex structure under much weaker assumptions.

5. Minimax inequalities

Motivated and inspired by the work of Chang and Yen [30], we introduce the following definition, which is more general than the well-known notions of quasiconcave and quasiconvex.

Definition 5.1. Let X be a nonempty set, K be a nonempty subset of X, and $(Y, D; \varphi_N)$ be an *FWC*-space. Let $f : X \times D \to \mathbb{R} \cup \{\pm\infty\}$ and $g : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be two functions. We say that g is *FW*-f-quasiconcave on Y with respect to K if for any $N = \{u_0, \dots, u_n\} \in \langle D \rangle$, any $\{u_{i_0}, \dots, u_{i_k}\} \subset \{u_0, \dots, u_n\}$ and for each $x \in K$, we have $g(x, y) \geq \min_{0 \leq j \leq k} f(x, u_{i_j})$ for all $y \in \varphi_N(\Delta_k)$. g is said to be *FW*-f-quasiconvex on Y with respect to K if $(-g)$ is *FW*-$(-f)$-quasiconcave on Y with respect to K. When $Y = D$ and $f = g$, f is said to be *FW*-quasiconcave (resp., *FW*-quasiconvex) on Y with respect to K. When $X = Y = K$, we say that g is *FW*-f-quasiconcave (resp., *FW*-quasiconvex) on the second X.

Remark 5.1. Definition 5.1 generalizes Definition 4.1 of [31] from *FC*-spaces to *FWC*-spaces. In our opinion, it is worth comparing Definition 5.1 with Definition 3 of [32].

Lemma 5.1. *Let X be a nonempty set, K be a nonempty subset of X, and $(Y, D; \varphi_N)$ be an FWC-space. Let $f : X \times D \to \mathbb{R} \cup \{\pm\infty\}$ and $g : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be two functions. Then the following two statements are equivalent:*

(i) *g is FW-f-quasiconcave (resp., FW-f-quasiconvex) on Y with respect to K;*
(ii) *for each $x \in K$ and each $\lambda \in \mathbb{R}$, the set $\{y \in Y : g(x, y) > \lambda\}$ (resp., $\{y \in Y : g(x, y) < \lambda\}$) is an FWC-subspace of $(Y, D; \varphi_N)$ relative to $\{u \in D : f(x, u) > \lambda\}$ (resp., $\{u \in D : f(x, u) < \lambda\}$).*

Proof. We only prove the conclusion for the case that g is *FW*-f-quasiconcave on Y with respect to K. The case that g is *FW*-f-quasiconvex on Y with respect to K is proved similarly.

(i) $\Rightarrow$ (ii). Suppose (i) is true. If (ii) does not hold, then there exist $x \in K$, $\lambda \in \mathbb{R}$, $N = \{u_0, \dots, u_n\} \in \langle D \rangle$ and $\{u_{i_0}, \dots, u_{i_k}\} \subset N \cap \{u \in D : f(x, u) > \lambda\}$ such that

$$\varphi_N(\Delta_k) \not\subset \{y \in Y : g(x, y) > \lambda\}.$$

Hence, there exists $y \in \varphi_N(\Delta_k)$ such that $y \notin \{y \in Y : g(x, y) > \lambda\}$, i.e., $g(x, y) \leq \lambda$. Since $\{u_{i_0}, \ldots, u_{i_k}\} \subset N \cap \{u \in D : f(x, u) > \lambda\}$, we have $f(x, u_{i_j}) > \lambda$ for each $j \in \{0, \ldots, k\}$. By (i), we obtain the following contradiction

$$\lambda \geq g(x, y) \geq \min_{0 \leq j \leq k} f(x, u_{i_j}) > \lambda.$$

Therefore, (ii) must hold.

(ii) $\Rightarrow$ (i). If the conclusion is false, then there exist $N = \{u_0, \ldots, u_n\} \in \langle D \rangle$, $\{u_{i_0}, \ldots, u_{i_k}\} \subset \{u_0, \ldots, u_n\}$, $x \in K$ and $y \in \varphi_N(\Delta_k)$ such that

$$g(x, y) < \min_{0 \leq j \leq k} f(x, u_{i_j}).$$

Let $g(x, y) < \lambda < \min_{0 \leq j \leq k} f(x, u_{i_j})$. Then we have $\{u_{i_0}, \ldots, u_{i_k}\} \subset N \cap \{u \in D : f(x, u) > \lambda\}$. From (ii), we get $\varphi_N(\Delta_k) \subset \{y \in Y : g(x, y) > \lambda\}$. Hence, $g(x, y) > \lambda > g(x, y)$. This is a contradiction. Therefore, (i) must be true. $\square$

Theorem 5.1. *Let X be a topological space, $(Y, D; \varphi_N)$ be an FWC-space, $T \in \widetilde{\mathfrak{B}}(Y, D, X)$ be a compact set-valued mapping, and $\lambda \in \mathbb{R}$. Let $e, f : X \times D \to \mathbb{R} \cup \{\pm\infty\}$ and $g : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be three functions such that*

(i) *$e(x, u)$ is λ-transfer compactly lower semicontinuous on X;*
(ii) *for each $(x, u) \in X \times D$, $e(x, u) \leq f(x, u)$;*
(iii) *g is FW-f-quasiconcave on Y with respect to $\overline{T(Y)}$;*
(iv) *for each $y \in Y$ and each $x \in T(y)$, $g(x, y) \leq \lambda$.*

Then there exists a point $\hat{x} \in \overline{T(Y)}$ such that $e(\hat{x}, u) \leq \lambda$ for each $u \in D$. In particular, we have $\inf_{x \in \overline{T(Y)}} \sup_{u \in D} e(x, u) \leq \lambda$.

Proof. In order to apply Theorem 3.1, let $L = \{(x, u) \in X \times D : e(x, u) \leq \lambda\}$, $P = \{(x, u) \in X \times D : f(x, u) \leq \lambda\}$ and $M = \{(x, y) \in X \times Y : g(x, y) \leq \lambda\}$. Define a set-valued mapping $A : D \to 2^X$ by $A(u) = \{x \in X : (x, u) \notin L\}$ for each $u \in D$. Since $A^c(u) = \{x \in X : (x, u) \in L\} = \{x \in X : e(x, u) \leq \lambda\}$ for each $u \in D$, we know that A^c is transfer compactly closed-valued by (i) and Lemma 2.1; thus, A is transfer compactly open-valued. By (ii), we have $P \subset L$. By (iii) and Lemma 5.1, for each $x \in \overline{T(Y)}$, the set $\{y \in Y : g(x, y) > \lambda\}$ is an *FWC*-subspace of $(Y, D; \varphi_N)$ relative to $\{u \in D : f(x, u) > \lambda\}$, i.e., for each $x \in \overline{T(Y)}$, the set $\{y \in Y : (x, y) \notin M\}$ is an *FWC*-subspace of $(Y, D; \varphi_N)$ relative to the set $\{u \in D : (x, u) \notin P\}$. By (iv), $\mathrm{Graph}(T^{-1}) \subset M$. Therefore, all the conditions of Theorem 3.1 are fulfilled. So, by Theorem 3.1, there exists a point $\hat{x} \in \overline{T(Y)}$ such that $\{\hat{x}\} \times D \subset L$. By the definition of L, there exists a point $\hat{x} \in \overline{T(Y)}$ such that $e(\hat{x}, u) \leq \lambda$ for each $u \in D$. In particular, we have $\inf_{x \in \overline{T(Y)}} \sup_{u \in D} e(x, u) \leq \lambda$. This completes the proof. $\square$

Remark 5.2. Theorem 5.1 improves and generalizes Theorem 3.1 of [33] from topological vector spaces to *FWC*-spaces without any linear and convex structure under much weaker assumptions.

Theorem 5.2. *Theorems* 3.1 and 5.1 *are equivalent.*

Proof. We have seen that Theorem 3.1 implies Theorem 5.1. Now, we prove that Theorem 5.1 implies Theorem 3.1. Suppose that all the conditions of Theorem 3.1 are satisfied. For each $(x, u, y) \in X \times D \times Y$, we define $e, f : X \times D \to \mathbb{R} \cup \{\pm\infty\}$ and $g : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ by

$$e(x, u) = \begin{cases} 0, & (x, u) \in L, \\ 1, & (x, u) \notin L, \end{cases} \qquad f(x, u) = \begin{cases} 0, & (x, u) \in P, \\ 1, & (x, u) \notin P, \end{cases}$$

$$g(x, y) = \begin{cases} 0, & (x, y) \in M, \\ 1, & (x, y) \notin M. \end{cases}$$

Let $\lambda = 0$. By the definition of e, for each $u \in D$, $\{x \in X : e(x, u) \leq 0\} = \{x \in X : (x, u) \in L\} = A^c(u)$. Then by (i) and Lemma 2.1, e is 0-transfer compactly lower semicontinuous on X. Since $P \subset L$, we have $e(x, u) \leq f(x, u)$ for each $(x, u) \in X \times D$. By the definitions of f and g, for each $x \in X$ and each $\lambda \in \mathbb{R}$, we have

$$\{y \in Y : g(x, y) > \lambda\} = \begin{cases} \{y \in Y : (x, y) \notin M\}, & 0 \leq \lambda < 1, \\ \emptyset, & \lambda \geq 1, \\ Y, & \lambda < 0, \end{cases}$$

$$\{u \in D : f(x, u) > \lambda\} = \begin{cases} \{u \in D : (x, u) \notin P\}, & 0 \leq \lambda < 1, \\ \emptyset, & \lambda \geq 1, \\ D, & \lambda < 0. \end{cases}$$

By (ii), we know that for each $x \in \overline{T(Y)}$ and each $\lambda \in \mathbb{R}$, the set $\{y \in Y : g(x, y) > \lambda\}$ is an *FWC*-subspace of $(Y, D; \varphi_N)$ relative to $\{u \in D : f(x, u) > \lambda\}$. Then by Lemma 5.1, g is *FW*-f-quasiconcave on Y with respect to $\overline{T(Y)}$. By (iii), for each $y \in Y$ and each $x \in T(y)$, $g(x, y) \leq 0$. Therefore, all of the requirements of Theorem 5.1 are satisfied, and hence, there exists a point $\hat{x} \in \overline{T(Y)}$ such that $e(\hat{x}, u) \leq 0$ for each $u \in D$, i.e., there exists a point $\hat{x} \in \overline{T(Y)}$ such that $\{\hat{x}\} \times D \subset L$. This completes the proof. $\square$

Theorem 5.3. *Let X be a topological space, $(Y, D; \varphi_N)$ be an FWC-space, $T \in \widetilde{\mathscr{B}}(Y, D, X)$ be a set-valued mapping, and $\lambda \in \mathbb{R}$. Let $e, f : X \times D \to \mathbb{R} \cup \{\pm\infty\}$ and $g : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be three functions such that*

(i) *$e(x, u)$ is λ-transfer compactly lower semicontinuous on X;*
(ii) *for each $(x, u) \in X \times D$, $e(x, u) \le f(x, u)$;*
(iii) *g is FW-f-quasiconcave on Y with respect to $\overline{T(Y)}$;*
(iv) *for each $y \in Y$ and each $x \in T(y)$, $g(x, y) \le \lambda$;*
(v) *there exists a compact subset K of X such that for each $x \in X \setminus K$, $g(x, y) > \lambda$ for all $y \in Y$.*

Then there exists a point $\hat{x} \in \overline{T(Y)}$ such that $e(\hat{x}, u) \le \lambda$ for each $u \in D$. In particular, we have $\inf_{x\in\overline{T(Y)}} \sup_{u\in D} e(x, u) \le \lambda$.

Proof. We show that T is compact. In fact, by (iv), we obtain the following

$$T(y) \subset \{x \in X : g(x, y) \le \lambda\} \quad \text{for each } y \in Y. \tag{5.1}$$

By (v), we have

$$\{x \in X : g(x, y) \le \lambda\} \subset K \quad \text{for each } y \in Y. \tag{5.2}$$

Combining (5.1) and (5.2), we deduce that $T(Y) \subset K$. Hence, T is compact. The rest of the proof is similar to that of Theorem 5.1. Thus, we can readily see that the conclusion of Theorem 5.3 holds. $\square$

Remark 5.3. Condition (v) of Theorem 5.3 is trivially fulfilled when X is compact.

Theorem 5.4. *Let X be a topological space, $(Y, D; \varphi_N)$ be an FWC-space, $T \in \widetilde{\mathscr{B}}(Y, D, X)$ be a compact set-valued mapping, and $\lambda \in \mathbb{R}$. Let $e, f : X \times D \to \mathbb{R} \cup \{\pm\infty\}$ and $g : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be three functions such that*

(i) *$e(x, u)$ is λ-transfer compactly lower semicontinuous on X;*
(ii) *for each $(x, u) \in X \times D$, $e(x, u) \le f(x, u)$;*
(iii) *g is FW-f-quasiconcave on Y with respect to $\overline{T(Y)}$.*

Then one of the following situations holds:

(a) *there exists $\widehat{x} \in \overline{T(Y)}$ such that $e(\widehat{x}, u) \le \lambda$ for all $u \in D$;*
(b) *there exists $(\widehat{y}, \widehat{x}) \in \mathrm{Graph}(T)$ such that $g(\widehat{x}, \widehat{y}) > \lambda$.*

Proof. Define three set-valued mappings $F : D \to 2^X$, $R : X \to 2^D$ and $\Omega : X \to 2^Y$ as follows:

$$F(u) = \{x \in X : e(x, u) \le \lambda\} \quad \text{for each } u \in D,$$
$$R(x) = \{u \in D : f(x, u) > \lambda\} \quad \text{for each } x \in X,$$
$$\Omega(x) = \{y \in Y : g(x, y) > \lambda\} \quad \text{for each } x \in X.$$

If (a) is false, then it follows that for each $x \in \overline{T(Y)}$, there exists a point $\widehat{u} \in D$ such that $e(x, \widehat{u}) > \lambda$, which implies that for each $x \in \overline{T(Y)}$, $F^*(x) \ne \emptyset$. By (i) and Lemma 2.1, F is transfer compactly closed-valued; thus, F^c is transfer compactly open-valued. By (ii), we have $\{x \in X : u \notin R(x)\} \subset F(u)$ for each $u \in D$. By (iii) and Lemma 5.1, we know that for each $x \in \overline{T(Y)}$, $\Omega(x)$ is an *FWC*-subspace of $(Y, D; \varphi_N)$ relative to $R(x)$. Therefore, all of the requirements of Theorem 4.2 are satisfied. Hence, by Theorem 4.2, there exists $(\widehat{x}, \widehat{y}) \in X \times Y$ such that $\widehat{x} \in T(\widehat{y})$ and $\widehat{y} \in \Omega(\widehat{x})$, i.e., there exists $(\widehat{y}, \widehat{x}) \in \mathrm{Graph}(T)$ such that $g(\widehat{x}, \widehat{y}) > \lambda$. This completes the proof. $\square$

Theorem 5.5. *Let X be a topological space, $(Y, D; \varphi_N)$ be an FWC-space, and $T \in \widetilde{\mathscr{B}}(Y, D, X)$ be a compact set-valued mapping. Let $e, f : X \times D \to \mathbb{R} \cup \{\pm\infty\}$ and $g : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be three functions such that*

(i) *$e(x, u)$ is transfer compactly upper semicontinuous on X;*
(ii) *for each $(x, u) \in X \times D$, $e(x, u) \ge f(x, u)$;*
(iii) *g is FW-f-quasiconvex on Y with respect to $\overline{T(Y)}$.*

Then there exists a point $\hat{x} \in \overline{T(Y)}$ such that $e(\hat{x}, u) \ge \inf_{y\in Y, x\in T(y)} g(x, y)$ for each $u \in D$. In particular, we have $\sup_{x\in\overline{T(Y)}} \inf_{u\in D} e(x, u) \ge \inf_{y\in Y, x\in T(y)} g(x, y)$.

Proof. We may assume that $\inf_{y\in Y, x\in T(y)} g(x, y) > -\infty$. Set $e' = -e$, $f' = -f$ and $g' = -g$, then e', f' and g' satisfy Conditions (i)–(iii) of Theorem 5.4. Let $\lambda = \sup_{y\in Y, x\in T(y)} g'(x, y)$. By the definition of λ, conclusion (b) of Theorem 5.4 does not hold. Hence, conclusion (a) of Theorem 5.4 holds. So, there exists $\widehat{x} \in \overline{T(Y)}$ such that $e'(\widehat{x}, u) \le \sup_{y\in Y, x\in T(y)} g'(x, y)$ for all $u \in D$, i.e., $e(\widehat{x}, u) \ge \inf_{y\in Y, x\in T(y)} g(x, y)$ for all $u \in D$. In particular, we have $\sup_{x\in\overline{T(Y)}} \inf_{u\in D} e(x, u) \ge \inf_{y\in Y, x\in T(y)} g(x, y)$. This completes the proof. $\square$

Remark 5.4. Theorem 5.5 generalizes Theorem 2.4 of [34] in several aspects, and in turn, it improves and generalizes Theorem 3 of [6], Theorem 1 of [35] and the well-known Ky Fan minimax inequality in [36] from topological vector spaces to *FWC*-spaces without any linear and convex structure under much weaker assumptions.

By taking $X = Y$ in Theorem 5.5, we obtain the following corollary.

Corollary 5.1. *Let X be a nonempty compact subspace of a topological space E and D be a nonempty set such that $(X, D; \varphi_N)$ is an FWC-space, and let $I_X \in \widetilde{\mathfrak{B}}(X, D, X)$, where I_X is identity map defined on X. Let $e, f : X \times D \to \mathbb{R} \cup \{\pm\infty\}$ and $g : X \times X \to \mathbb{R} \cup \{\pm\infty\}$ be three functions such that*

(i) *$e(x, u)$ is transfer compactly upper semicontinuous on X;*
(ii) *for each $(x, u) \in X \times D$, $e(x, u) \geq f(x, u)$;*
(iii) *g is FW-f-quasiconvex on the second X.*

Then there exists a point $\hat{x} \in X$ such that $e(\hat{x}, u) \geq \inf_{x\in X} g(x, x)$ for each $u \in D$. In particular, we have $\sup_{x\in X} \inf_{u\in D} e(x, u) \geq \inf_{x\in X} g(x, x)$.

Remark 5.5. Corollary 5.1 generalizes Corollary 4.2 of [31] from *FC*-spaces to *FWC*-spaces without any linear and convex structure.

Theorem 5.6. *Let X be a topological space, $(Y, D; \varphi_N)$ be an FWC-space, and let $e, f : X \times D \to \mathbb{R} \cup \{\pm\infty\}$ and $g, h : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be four functions. Let $\beta = \inf_{K\in\mathcal{C}(X)} \sup_{y\in Y} \inf_{x\in K} h(x, y)$, where $\mathcal{C}(X)$ is the family of the nonempty compact subsets of X. Suppose the following conditions are fulfilled:*

(i) *for each $\lambda > \beta$, $e(x, u)$ is λ-transfer compactly lower semicontinuous on X;*
(ii) *for each $(x, u, y) \in X \times D \times Y$, $e(x, u) \leq f(x, u)$ and $g(x, y) \leq h(x, y)$;*
(iii) *for each $K \in \mathcal{C}(X)$, g is FW-f-quasiconcave on Y with respect to K;*
(iv) *for each $K \in \mathcal{C}(X)$ and each $\lambda > \beta$, the set-valued mapping $y \to \{x \in K : h(x, y) \leq \lambda\}$ belongs to $\widetilde{\mathfrak{B}}(Y, D, X)$.*

Then we have $\inf_{x\in\overline{T(Y)}} \sup_{u\in D} e(x, u) \leq \inf_{K\in\mathcal{C}(X)} \sup_{y\in Y} \inf_{x\in K} h(x, y)$. If X is compact, then we have

$$\inf_{x\in\overline{T(Y)}} \sup_{u\in D} e(x, u) \leq \sup_{y\in Y} \inf_{x\in X} h(x, y).$$

Proof. If $\beta = +\infty$, then the theorem is obviously true. Hence, we may assume that $\beta < +\infty$. Now, we let $\lambda > \beta$ be fixed and then let $L = \{(x, u) \in X \times D : e(x, u) \leq \lambda\}$, $P = \{(x, u) \in X \times D : f(x, u) \leq \lambda\}$ and $M = \{(x, y) \in X \times Y : g(x, y) \leq \lambda\}$. Define a set-valued mapping $A : D \to 2^X$ by $A(u) = \{x \in X : (x, u) \notin L\}$ for each $u \in D$. Since $A^c(u) = \{x \in X : (x, u) \in L\} = \{x \in X : e(x, u) \leq \lambda\}$ for each $u \in D$, we know that A^c is transfer compactly closed-valued by (i) and Lemma 2.1; thus, A is transfer compactly open-valued. By the first part of (ii), it is easy to verify that $P \subset L$. Let K be a nonempty compact subset of X such that

$$\lambda > \sup_{y\in Y} \inf_{x\in K} h(x, y). \tag{5.3}$$

Define a set-valued mapping $T : Y \to 2^K$ by $T(y) = \{x \in K : h(x, y) \leq \lambda\}$ for each $y \in Y$. Clearly, $T(Y) \subset K$, so T is compact. By the second part of (ii), $\mathrm{Graph}(T^{-1}) \subset M$. By (iii) and Lemma 5.1, for each $x \in \overline{T(Y)}$, the set $\{y \in Y : g(x, y) > \lambda\}$ is an *FWC*-subspace of $(Y, D; \varphi_N)$ relative to $\{u \in D : f(x, u) > \lambda\}$, i.e., for each $x \in \overline{T(Y)}$, the set $\{y \in Y : (x, y) \notin M\}$ is an *FWC*-subspace of $(Y, D; \varphi_N)$ relative to the set $\{u \in D : (x, u) \notin P\}$. By (iv) and (5.3), $T \in \widetilde{\mathfrak{B}}(Y, D, X)$ and $T(y) \neq \emptyset$ for each $y \in Y$. Hence, all the conditions of Theorem 3.1 are fulfilled. Therefore, by Theorem 3.1, there exists a point $\hat{x} \in \overline{T(Y)} \subset K$ such that $\{\hat{x}\} \times D \subset L$, that is, $e(\widehat{x}, u) \leq \lambda$ for all $u \in D$. Thus, we have

$$\inf_{x\in\overline{T(Y)}} \sup_{u\in D} e(x, u) \leq \lambda.$$

Since the above inequality holds for any $\lambda > \beta$, we simply let λ decrease to β to obtain the conclusion that $\inf_{x\in\overline{T(Y)}} \sup_{u\in D} e(x, u) \leq \inf_{K\in\mathcal{C}(X)} \sup_{y\in Y} \inf_{x\in K} h(x, y)$. If X is compact, then we have $\inf_{x\in\overline{T(Y)}} \sup_{u\in D} e(x, u) \leq \sup_{y\in Y} \inf_{x\in X} h(x, y)$. This completes the proof. □

Remark 5.6. Theorem 5.6 generalizes Theorem 2.2 of [34] in the following aspects: (a) from *H*-spaces to *FWC*-spaces; (b) from three functions to four functions; (c) Conditions (i) and (iii) of Theorem 5.6 are weaker than Conditions (ii) and (iii) of Theorem 2.2 in [34], respectively; (d) Condition (iv) of Theorem 2.2 in [34] is removed.

From Theorem 5.6, we have the following corollary.

Corollary 5.2. *Let X be a topological space, $(Y; \varphi_N)$ be an FWC-space, and let $f : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be a function. Let $\beta = \inf_{K\in\mathcal{C}(X)} \sup_{y\in Y} \inf_{x\in K} f(x, y)$, where $\mathcal{C}(X)$ is the family of the nonempty compact subsets of X. Suppose the following conditions are fulfilled:*

(i) *for each $\lambda > \beta$, $f(x, y)$ is λ-transfer compactly lower semicontinuous on X;*
(ii) *for each $K \in \mathcal{C}(X)$, f is FW-quasiconcave on Y with respect to K;*
(iii) *for each $K \in \mathcal{C}(X)$ and each $\lambda > \beta$, the set-valued mapping $y \to \{x \in K : f(x, y) \leq \lambda\}$ belongs to $\widetilde{\mathfrak{B}}(Y, X)$.*

Then we have $\inf_{x\in\overline{T(Y)}} \sup_{y\in Y} f(x, y) = \inf_{K\in\mathcal{C}(X)} \sup_{y\in Y} \inf_{x\in K} f(x, y)$. If X is compact, then we have

$$\inf_{x\in\overline{T(Y)}} \sup_{y\in Y} f(x, y) = \sup_{y\in Y} \inf_{x\in X} f(x, y).$$

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